Teorema Della Differenziabilità Totale

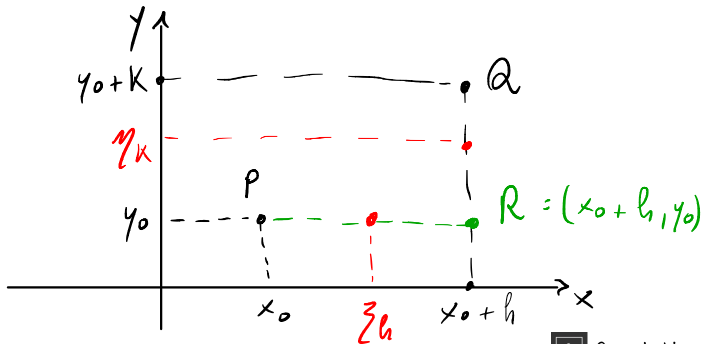
Se $f : A(\subseteq \mathbb{R}^2) \to \mathbb{R}$, con A aperto, una funzione con derivate parziali $f_x(x, y)$ e $f_y(x, y)$ continue in $(x_0, y_o) \in A$.

Allora f è differenziabile in $(x_0, y_o) \in A$.

Dim

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f_x(x_0,y_0)h - f_y(x_0,y_0)k - f(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$$

$$\underbrace{f(x_0+h, y_0+k)}_{f(Q)} - \underbrace{f(x_0,y_0)}_{f(P)}$$



$$f(Q) - f(P) = f(Q) - f(R) + f(R) - f(P) =$$

$$= f(x_0 + h, y_0 + K) - f(x_0 + h, y_0) + f(x_0 + h, y_0) - f(x_0, y_0)$$

Thm Lagrange

$\boxed{\text{Sia } f \text{ continua } [a,b] \text{ e derivabile in } (a,b). \ \exists\, c \in (a,b) \text{ t.c. } f'(c) = \dfrac{f(b)-f(a)}{b-a}}$

- $f(x_0 + h, y_0 + K) - f(x_0 + h, y_0) = K f_y(x_0 + h, \eta_K)$
- $f(x_0 + h, y_0) - f(x_0, y_0) = h f_x(\xi_h, y_0)$

$\Downarrow$ Sommando membro a membro

$$f(x_0 + h, y_0 + K) - f(x_0, y_0) = h f_x(\xi_h, y_0) + K f_y(x_0 + h, \eta_K)$$

$$\frac{\overbrace{f(x_0 + h, y_0 + K) - f(x_0, y_0)} - f_x(x_0, y_0) h - f_y(x_0, y_0) K}{\sqrt{h^2 + K^2}} =$$

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$$= \frac{h\, f_x(\xi_h, y_0) + K\, f_y(x_0+h, \eta_K) - f_x(x_0, y_0)\, h - f_y(x_0, y_0)\, K}{\sqrt{h^2+K^2}} =$$

$$= \frac{[f_x(\xi_h, y_0) - f_x(x_0, y_0)]\, h + K\, [f_y(x_0+h, \eta_K) - f_y(x_0, y_0)]}{\sqrt{h^2+K^2}} \Rightarrow$$

$\Rightarrow$ studiamone il modulo:

$$\left| \frac{[f_x(\xi_h, y_0) - f_x(x_0, y_0)]\, h + K\, [f_y(x_0+h, \eta_K) - f_y(x_0, y_0)]}{\sqrt{h^2+K^2}} \right| =$$

$$= \frac{\left|\left[f_x(\xi_h, y_0) - f_x(x_0, y_0)\right]h + k\left[f_y(x_0+h, \eta_k) - f_y(x_0, y_0)\right]\right|}{\sqrt{h^2+k^2}} \le$$

$$|a+b| \le |a| + |b|$$

$$\le \frac{\left|\left[f_x(\xi_h, y_0) - f_x(x_0, y_0)\right]h\right| + \left|k\left[f_y(x_0+h, \eta_k) - f_y(x_0, y_0)\right]\right|}{\sqrt{h^2+k^2}} =$$

$$= \frac{\left|\left[f_x(\xi_h, y_0) - f_x(x_0, y_0)\right]\right||h| + |k|\left|\left[f_y(x_0+h, \eta_k) - f_y(x_0, y_0)\right]\right|}{\sqrt{h^2+k^2}} =$$

$$= \frac{|f_x(\xi_h, y_0) - f_x(x_0,y_0)|\,|h|}{\sqrt{h^2+k^2}} + \frac{|f_y(x_0+h, \eta_k) - f_y(x_0,y_0)|\,|k|}{\sqrt{h^2+k^2}} \leq$$

$$\leq |f_x(\xi_h, y_0) - f_x(x_0,y_0)| + |f_y(x_0+h, \eta_k) - f_y(x_0,y_0)| \to 0 \quad \text{per } (h,k)\to(0,0)$$

$$(h,k) \to (0,0) \quad \Rightarrow \quad \begin{array}{l} (\xi_h, y_0) \to (x_0,y_0) \\ (x_0+h, \eta_k) \to (x_0,y_0) \end{array}$$ in quanto pti interni