Proposizione 1

Se $f : A(\subseteq \mathbb{R}^2) \to \mathbb{R}$ è differenziabile in $\mathbf{x}_0 \in A$, allora f è continua in $\mathbf{x}_0$.

Proposizione 2

Se $f : A(\subseteq \mathbb{R}^2) \to \mathbb{R}$ è differenziabile in $\mathbf{x}_0 \in A$, allora f è ammette in $\mathbf{x}_0$ le derivate direzionali lungo un qualunque vettore $\mathbf{v} \neq \mathbf{0}$ e si ha

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = \langle \nabla \mathbf{f}(\mathbf{x_0}), \mathbf{v} \rangle.$$

Dim 1

$\underset{\sim}{x}_0 \in A \equiv (x_0, y_0) \in A$

$$\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} \frac{f(\underset{\sim}{x}) - \underset{\sim}{\nabla} f(\underset{\sim}{x}_0)(\underset{\sim}{x} - \underset{\sim}{x}_0) - f(\underset{\sim}{x}_0)}{\| \underset{\sim}{x} - \underset{\sim}{x}_0 \|} = 0$$

Oss

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0+h, y_0+k) - f_x(x_0,y_0)h - f_y(x_0,y_0)k - f(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$$

$$\begin{aligned} x &= x_0 + h \\ y &= y_0 + k \end{aligned} \Rightarrow \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix}$$

$$f(\underset{\sim}{x}) = f(\underset{\sim}{x}_0) + \underset{\sim}{\nabla} f(\underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(\| \underset{\sim}{x} - \underset{\sim}{x}_0 \|)$$

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$$

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \lim_{\underline{x} \to \underline{x}_0} \left[f(\underline{x}_0) + \underline{\nabla} f(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) + o\left(\|\underline{x} - \underline{x}_0\|\right)\right] =$$

$$= f(\underline{x}_0)$$

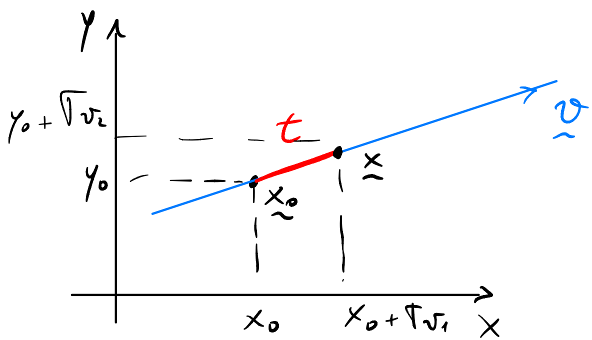
$$\boxed{\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)} \Rightarrow f \text{ è continua.}$$

Gquadroblog

Dim 2

$$f(\underset{\sim}{x}) = f(\underset{\sim}{x}_0) + \underset{\sim}{\nabla} f(\underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(\|\underset{\sim}{x} - \underset{\sim}{x}_0\|) \quad \text{per } \underset{\sim}{x} \to \underset{\sim}{x}_0$$

$$\underset{\sim}{x} = \underset{\sim}{x}_0 + t\,\underset{\sim}{v}$$



$$\lim_{t \to 0} \frac{f(\underset{\sim}{x}_0 + t\,\underset{\sim}{v}) - f(\underset{\sim}{x}_0)}{t} = \langle \underset{\sim}{\nabla} f(\underset{\sim}{x}_0), \underset{\sim}{v} \rangle$$

$$f(\underset{\sim}{x}_0 + t\,\underset{\sim}{v}) = f(\underset{\sim}{x}_0) + \underset{\sim}{\nabla} f(\underset{\sim}{x}_0) \cdot (\cancel{\underset{\sim}{x}_0} + t\,\underset{\sim}{v} - \cancel{\underset{\sim}{x}_0}) + o(\|\cancel{\underset{\sim}{x}_0} + t\,\underset{\sim}{v} - \cancel{x_0}\|)$$

$$= f(\underset{\sim}{x_0}) + t\, \underset{\sim}{\nabla} f(\underset{\sim}{x_0}) \cdot \underset{\sim}{v} + o(\|t\, \underset{\sim}{v}\|) \qquad \|t\, \underset{\sim}{v}\| \to 0$$

Poiché $\|t\, \underset{\sim}{v}\| = |t|\, \|\underset{\sim}{v}\|$ si ha $o(\|t\, \underset{\sim}{v}\|) = o(t)$

$$\lim_{t \to 0} \frac{f(\underset{\sim}{x_0} + t\, \underset{\sim}{v}) - f(\underset{\sim}{x_0})}{t} = \lim_{t \to 0} \frac{\cancel{f(\underset{\sim}{x_0})} + t\, \underset{\sim}{\nabla} f(\underset{\sim}{x_0}) \cdot \underset{\sim}{v} + o(t) - \cancel{f(\underset{\sim}{x_0})}}{t} =$$

$$= \lim_{t \to 0} \frac{t\, \underset{\sim}{\nabla} f(\underset{\sim}{x_0}) \cdot \underset{\sim}{v} + o(t)}{t} = \underset{\sim}{\nabla} f(\underset{\sim}{x_0}) \cdot \underset{\sim}{v}$$